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# Optimization method for an evolutional type inverse heat conduction problem

**Zui-Cha Deng, Jian-Ning Yu and Liu Yang**

Department of Mathematics, Lanzhou Jiaotong University Lanzhou, Gansu 730070, People's Republic of China

E-mail: zc_deng78@hotmail.com, yujn@mail.lzjtu.cn and l_yang218@163.com

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**Abstract**

This paper deals with the determination of a pair $(q, u)$ in the heat conduction equation

$$u_t - u_{xx} + q(x, t)u = 0,$$

with initial and boundary conditions

$$u(x, 0) = u_0(x), \qquad u_x|_{x=0} = u_x|_{x=1} = 0,$$

from the overspecified data $u(x, t) = g(x, t)$. By the time semi-discrete scheme, the problem is transformed into a sequence of inverse problems in which the unknown coefficients are purely space dependent. Based on the optimal control framework, the existence, uniqueness and stability of the solution $(q, u)$ are proved. A necessary condition which is a couple system of a parabolic equation and parabolic variational inequality is deduced.

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## 1. Introduction

In this paper, we study an evolutional type inverse problem of recovering the radiative coefficient of heat conduction equation from some additional conditions, which has important application in a large field of applied science. The problem can be stated in the following form:

**Problem P.** *Consider the following heat conduction equation:*

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - q(x, t)u, \qquad (x, t) \in Q = (0, 1) \times (0, T] \tag{1.1}$$

$$u(x, 0) = u_0(x), \qquad x \in (0, 1) \tag{1.2}$$

$$u_x(0, t) = u_x(1, t) = 0, \qquad t \in (0, T] \tag{1.3}$$

*where* $u_0(x)$ *is a given smooth function on interval* $(0, 1)$, $q(x,t)$ *is an unknown coefficient in (1.1). Assume that*

$$u(x,t) = g(x,t), \qquad x \in (0,1) \tag{1.4}$$

*is given. Determine the functions u and q satisfying (1.1)–(1.3).*

In physics, this model (1.1) describes the heat conduction procedure in a given homogeneous medium $Q$. The initial condition (1.2) means that at $t = 0$, there exists an initial temperature distribution $u_0(x)$ on interval $(0, 1)$, while the homogeneous Neumann boundary condition (1.3) means that there is no heat exchange with outside from $t = 0$ to $t = T$, i.e., the heat conductor is kept heat insulating all the time. The unknown coefficient $q(x,t)$ is called the radiative coefficient which is often related to the medium property. If the medium is inhomogeneous with some input source $f(x,t)$, then the equation should be written as

$$u_t - \nabla(p(x)\nabla u) + q(x,t)u = f(x,t), (x,t) \in Q.$$

The coefficient $p(x)$ represents the heat conduction property such as heat capacity, while $q(x,t)u(x,t)$ is also the heat source which depends both on location $x$, time $t$ and temperature $u$, except for the heat source $f(x,t)$. In fact, $q(x,t)$ describes the medium property of generating heat source or heat sink (see [4]).

The tasks of inverse heat problems are detection of heat conduction properties of the medium from some information of the solution, i.e., the determination of the unknown coefficient(s) in the heat equation from some additional information about $u(x,t)$. However, there is a fundamental difference between the direct and the inverse problem. It is well known that in all cases the inverse problem is ill-posed or improperly posed in the sense of Hardamard, while the direct problem is well-posed (see [18, 23]). The ill-posedness, especially the numerical instability, is a main difficulty for problem **P**. Since there always exist inevitable errors in the extra condition $g(x,t)$ which is often obtained by experiments, a small perturbation in $g(x,t)$ may result in a big change in $q(x,t)$, which may make the obtained results meaningless (see [26, 30]).

Inverse coefficient problems for parabolic equations are well studied in the literature. The inverse heat conduction problems using final temperature as inversion input data have been considered carefully, see [14, 22] for determining $p(x)$ in the above equation with $q(x,t) = 0$ from the measurement given $u(x,T)$. In this model, the heat conduction procedure is considered only for the input linear source $f(x,t)$, ignoring the nonlinear source $q(x,t)u(x,t)$ within the medium. In [19, 20], the inverse problem of identifying the principle coefficient $a(x)$ in 1D equation

$$u_t - a(x)u_{xx} + b(x)u_x + c(x)u = f(x,t)$$

from final overdetermination data $u(x,t_0)$ has been studied carefully by using an optimal control framework. The existence of $a(x)$ and a well-posed algorithm are obtained. Recently, the authors of [7, 21, 24] consider an evolutional type inverse problem and derive a stability result for the case $a = a(x,t)$. In [11], an inverse problem of identifying the local volatility from market prices of options is resolved by Tikhonov regularization method. The uniqueness and stability of determining $a(x)$ in the parabolic equation

$$\partial_t u + \mathcal{A}u + a(x)u = 0$$

from the final measurement data is considered in [17], where zero initial condition and nonzero boundary condition $\mathcal{B}u = \phi$ in $\partial Q \times (0,T)$ are assumed. The theoretical issues such as existence and uniqueness of coefficients inversion for parabolic equation are also studied in [18, 27]. In [16], the inverse problem of identification of a discontinuous source term $\mathcal{X}_D$ in the parabolic equation

$$\partial_t u - \triangle u = \mathcal{X}_D \qquad \text{in} \qquad \Omega \times (0, T)$$

from information on the flux is investigated, where $D$ is a domain constraint in the domain $\Omega \in \mathscr{R}^2$.

The purely space dependent case $q = q(x)$ has been investigated by several authors, e.g., in [4, 6, 28, 29, 31]. In [6], Hölder space method is applied to determine the unknown coefficient $q(x)$ from additional information given at $t = T$. Existence and uniqueness for the determination of $q(x)$ are derived in [28] by using the contracting mapping principle. In [4, 31], motivated by heuristic arguments, the optimization method is applied to stabilize the inverse problem. The authors of [4] prove the existence of minimizer and the convergence of approximate solution in finite-dimensional space, while in [31], the authors construct a new control functional and prove the existence and uniqueness of minimizer.

The case of purely time dependent $q = q(t)$ has been extensively studied by several authors (see, for instance, [2, 3, 8–10, 25]).

Finally, we note that the inverse problems of determining a source term $F(x, t)$ in the following parabolic equation:

$$u_t = (k(x, t)u_x)_x + F(x, t), \qquad (x, t) \in \Omega_T = (0, 1) \times (0, T)$$

have received considerable attention in the literature (see, for instance, [1, 15, 32]).

In this paper, we use an optimal control framework [19, 20, 31] to discuss problem **P** mainly from the theoretical analysis angle (see [4] which also used the optimal control method, but focused more on numerical computations). Being different from the problem in [4, 31], where the function $q$ is purely space dependent, i.e., $q = q(x)$, the unknown coefficient $q$ in this paper not only depends on the space variable $x$, but also depends on the time $t$, i.e., $q = q(x, t)$. The methods used in [4, 31] are not applicable for problem **P**, for the reason that if one attempts to recover $q(x, t)$ from the extra condition (1.4) by optimization method as he did in [4, 31], i.e., construct a control functional

$$J = J(q(x, t))$$

and minimize it to obtain $q(x, t)$, then it will be quite difficult for one to find an appropriate form of the control functional of which the minimum is stable. However, in [31] we obtain that if an extra condition

$$u(x, T) = f(x), \qquad x \in (0, 1)$$

is given, where $f(x)$ is a given function, then the unknown coefficient $q(x)$ can be identified uniquely and stably under the condition that $T$ is small enough (see figure 1). Motivated by the idea in [31], we find a way to reconstruct the unknown coefficient $q(x, t)$ for problem **P**. We solve it by using semi-discrete scheme, i.e., we find $q(x, t_n)$ step by step, where $t_n = nh$ and $h = \frac{T}{N}, n = 0, 1, \ldots, N$. In fact, if $q(x, t_0), \ldots, q(x, t_{n-1})$ has been defined, then from a known extra condition

$$u(x, t_n) = g(x, t_n),$$

we find $q(x, t_n)$ such that

$$J_n(q(\cdot, t_n)) = \inf_{q \in \mathcal{A}} J_n(q),$$

where $\mathcal{A}$ is an appropriately admissible set and $J_n(q)$ is a control functional (see figure 2).

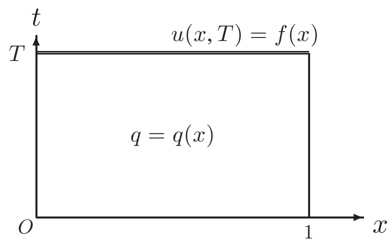


**Figure 1.** The recovery of $q(x)$.

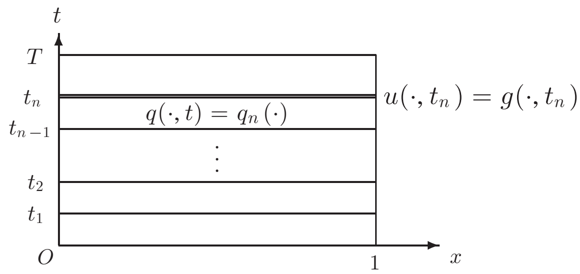


**Figure 2.** The recovery of $q_n(x)$.

As indicated in [31], if $h$ is small enough, $q(x,t_n)$ can be identified uniquely and stably. Thus, for any $h$ we obtain an approximate function $q^h(x,t)$ defined as follows:

$$q^h(x,t) = \begin{cases} q(x,t_n), & t = t_n, \\ \text{linear}, & t_{n-1} \leqslant t \leqslant t_n. \end{cases}$$

In the sense of numerical computation, $q^h(x,t)$ can be taken as an approximate solution of $q(x,t)$. Since the main purpose of this paper is to discuss the inverse problem from the theoretical analysis angle, we then thoroughly analyse the asymptotic behavior of $q^h(x,t)$ as $h \to 0$.

The paper is organized as follows. In section 2, problem **P** is transformed into a sequence of optimal control problems $P_n$ and the necessary condition which must be satisfied by the minimum of problem $P_n$ is obtained. In order to discuss the asymptotic behavior of $q^h(x,t)$, we establish some uniform estimates for the approximate solution in section 3. In section 4, we prove that there exists a subsequence of $q^h(x,t)$ which converge to a function $q(x,t)$ and deduce the necessary condition which must be satisfied by $q(x,t)$. Finally, the uniqueness and stability of $q(x,t)$ are obtained.

## 2. Time semi-discrete scheme

The well-known Schauder theory for parabolic equations guarantees that, for any given positive coefficient $q(x,t) \in C^{\alpha,\frac{\alpha}{2}}(\bar{Q})$, there exists a unique solution $u(x,t) \in C^{2+\alpha,1+\frac{\alpha}{2}}(\bar{Q})$, to equation (1.1)–(1.3).

Assume that additional condition (1.4) satisfies

$$g \in C^{\alpha,\frac{\alpha}{2}}(\bar{Q}), \tag{2.1}$$

and there exists a constant $C$ such that

$$\|g\|_{C^{\alpha,\frac{\alpha}{2}}(\bar{Q})} \leqslant C, \qquad \max_{0\leqslant t\leqslant T} \|g(\cdot,t)\|_{H^1(0,1)} \leqslant C. \tag{2.2}$$

**Remark 2.1.** The extra condition (1.4) is essential for the inverse problem **P**. It means that continuous observation of $u(x,t)$ is available on $\bar{Q}$, i.e.,

$$u(x,t) = g(x,t), \qquad (x,t) \in \bar{Q},$$

where $g(x,t)$ is a known function. The extra condition (1.4) facilitates the theoretical analysis, while in practice it is not applicable. In fact, it is impossible to measure the temperature $u(x,t)$ at every position $x$ and every time $t$. The observations of $u(x,t)$ can only be given as the discrete data, e.g., an appropriate form is

$$u(x_i,t_n) = g(x_i,t_n), \qquad i = 1,2,\cdots,M; \quad j = 1,2,\cdots,N; \, (x_i,t_n) \in \bar{Q}, \tag{2.3}$$

where $M$ and $N$ are two positive integers. Therefore, additional condition (1.4) and the regularity of $g(x,t)$ (2.1)/(2.2) are obtained from (2.3) by the interpolation and smoothing technique. In such case, $g(x,t)$ is not the exact solution of (1.1)–(1.3) but the 'artificial' solution which surely contains errors.

It should be pointed out that if $g(x,t)$ is indeed the exact solution of (1.1)–(1.3), then the unknown function $q(x,t)$ can be found directly from equation (1.1) by the following formula:

$$q(x,t) = \frac{g_{xx} - g_t}{g}, \tag{2.4}$$

provided $g(x,t) \neq 0$, $(x,t) \in Q$. However, formula (2.4) is not applicable for $g(x,t)$ which contains errors. Since we have to compute the numerical derivatives of $g(x,t)$ with respect to $x$ and $t$, particularly the second derivative with respect to $x$, arbitrarily small changes in $g$ may lead to arbitrarily large changes in its derivative, which may make the obtained $q$ meaningless. Moreover, for the inexact data $g(x,t)$ it is difficult to guarantee that there exists at least a solution to the original inverse problem, thus the optimization technique should be applied to get some general solutions.

To reconstruct the unknown coefficient, we introduce the following time semi-discrete cost functional and time semi-discrete optimal control problem.

Let

$$0 = t_0 < t_1 < \cdots < t_N = T$$

be a partition of interval $[0,T]$ with $t_n = nh$ and $h = \frac{T}{N}$. Let

$$\mathcal{A} = \{q(x) | 0 < \alpha_0 \leqslant q \leqslant \alpha_1, q \in H^1(0,1)\}$$

be the admissible set, where $\alpha_0$ and $\alpha_1$ are given positive constants.

Beginning with a given function $q_0 \in \mathcal{A}$ with

$$q_0 \in W^{1,\infty}(0,1),$$

we introduce the following optimal control problem:

**Problem** $P_n$: *assume that* $q_0, q_1, \cdots, q_{n-1} \in \mathcal{A}$ *are known. Find* $q_n \in \mathcal{A}$ *such that*

$$J_n(q_n) = \min_{q\in\mathcal{A}} J_n(q), \tag{2.5}$$

where

$$J_n(q) = \frac{\sigma}{2}\left(\frac{1}{h}\|q - q_{n-1}\|^2_{L^2(0,1)} + \|\nabla q\|^2_{L^2(0,1)}\right) + \frac{1}{2h}\|u(\cdot,t_n;q) - g(\cdot,t_n)\|^2_{L^2(0,1)}, \tag{2.6}$$

$u(x, t; q)$ is the solution of (1.1)–(1.3) in $[0, t_n]$ corresponding to coefficient

$$\tilde{q} = \begin{cases} \dfrac{t - t_{n-1}}{h} q(x) + \dfrac{t_n - t}{h} q_{n-1}(x), & t_{n-1} \leqslant t \leqslant t_n, \\ \dfrac{t - t_{k-1}}{h} q_k(x) + \dfrac{t_k - t}{h} q_{k-1}(x), & t_{k-1} \leqslant t \leqslant t_k, 1 \leqslant k \leqslant n-1, \end{cases} \tag{2.7}$$

and $\sigma > 0$ is a regularization parameter.

**Remark 2.2.** The requirement on $q(x, t) > 0$ is not essential. For $q(x, t)$ with lower bound $c_0 < 0$, we can use the transform $v(x, t) = u(x, t)\,\mathrm{e}^{(c_0-1)t}$ , which satisfies

$$v_t - v_{xx} + (q(x, t) - c_0 + 1)v = 0.$$

Then we have

$$q(x, t) - c_0 + 1 > 0.$$

So the same kind of inverse problem for function $v(x, t)$ is constituted with $q(x, t) - c_0 + 1 > 0$. We can use the optimization technique proposed in this paper to recover $q(x, t) - c_0 + 1$.

**Theorem 2.1.** *There exists a minimizer $q_n \in \mathcal{A}$ of $J(q)$, i.e.*

$$J_n(q_n) = \min_{q \in \mathcal{A}} J_n(q).$$

The proof of this theorem is available in [31].

Such a $q_n$ is called an optimal control of problem $P_n$. From this theorem, the functions $q_0, q_1, \ldots, q_N \in \mathcal{A}$ are well defined when $q_0 \in \mathcal{A}$ is given. For $(x, t) \in \bar{Q}$, let

$$q^h(x, t) = \frac{t - t_{n-1}}{h} q_n(x) + \frac{t_n - t}{h} q_{n-1}(x), \qquad t_{n-1} \leqslant t \leqslant t_n, \quad n = 1, \cdots, N.$$

$q^h(x, t)$ is called the discrete reconstruction of unknown coefficient. Then recovering the unknown coefficient is reduced to finding the sequence of optimal control and investigating the behavior of the sequence of optimal control and the discrete reconstruction $q^h(x, t)$ as $h \to 0$.

**Remark 2.3.** By the local uniqueness obtained in [31], with a given $q_0 \in \mathcal{A}$, the coefficients $q_1, \ldots, q_N$ can be uniquely identified as $\frac{\sqrt{h}}{\sigma} \to 0$. Then the uniqueness of $q^h(x, t)$ is the direct deduction of the uniqueness of $q_n(x)$, $n = 0, 1, \ldots, N$.

**Remark 2.4.** Without the first item $\frac{\sigma}{2h}\|q - q_{n-1}\|^2_{L^2(0,1)}$ in (2.6), the control functional $J_n$ will be same to that constructed in [31]. For such $J_n$, its minimizer $q_n$ can also be recovered uniquely and stably. However, it may be very difficult to illustrate the smoothness of the discrete reconstruction $q^h(x, t)$, especially the smoothness of the limiting function of $q^h(x, t)$ as $h \to 0$, which is essential to obtain the stability of the limiting function. Motivated by the creative work in [21] where an evolutional type inverse problem arisen in finance is resolved completely, we also add the cost item $\frac{\sigma}{2h}\|q - q_{n-1}\|^2_{L^2(0,1)}$ to the control functional $J_n$ to stabilize the inverse problem. Other approaches to construct the control functional are available in [7, 24].

Now we derive the necessary condition for the optimal control problem $P_n$ as follows:

**Theorem 2.2.** *Assume that $q_0 \in \mathcal{A}$ is given. Let $q_n \in \mathcal{A}$ be an optimal control of problem $P_n$, $n = 1, \ldots, N$ and $u^h(x, t)$ be the solution of (1.1)–(1.3) in $[0, T]$ corresponding to coefficient $\tilde{q} = q^h(x, t)$. Then we have, for any $\omega \in \mathcal{A}$*

$$\sigma \int_0^1 \left[ \frac{q_n - q_{n-1}}{h}(\omega - q_n) + \nabla q_n \cdot \nabla(\omega - q_n) \right] \mathrm{d}x + \frac{1}{h} \int_{t_{n-1}}^{t_n} \int_0^1 \frac{t - t_{n-1}}{h}(q_n - \omega) u^h v^h \, \mathrm{d}x \, \mathrm{d}t \geqslant 0, \tag{2.8}$$

*where $v^h(x, t)$ satisfies the following equation:*

$$\begin{aligned} & -v_t^h - v_{xx}^h + q^h(x, t) v^h = 0, \qquad (x, t) \in (0, 1) \times [t_{n-1}, t_n] \\ & v_x^h(0, t) = v_x^h(1, t) = 0, \\ & v^h(x, t_n) = u^h(x, t_n) - g(x, t_n). \end{aligned} \tag{2.9}$$

**Proof.** Let $q_n \in \mathcal{A}$ be an optimal control of problem $P_n$. Note that $\mathcal{A}$ is a convex set, for any $\omega \in \mathcal{A}$,

$$q^\lambda = (1 - \lambda) q_n + \lambda \omega \in \mathcal{A}, \qquad \lambda \in [0, 1].$$

Hence for any $\omega \in \mathcal{A}$, the function $j(\lambda) = J_n(q^\lambda)$ is well defined and reaches its minimum at $\lambda = 0$. Then we have

$$j'(0) = \left. \frac{\mathrm{d}}{\mathrm{d}\lambda} J_n(q^\lambda) \right|_{\lambda=0} \geqslant 0,$$

i.e., for any $\omega \in \mathcal{A}$,

$$\left. \frac{\mathrm{d}}{\mathrm{d}\lambda} \int_0^1 \left[ \sigma \left( \frac{|q^\lambda - q_{n-1}|^2}{h} + |\nabla q^\lambda|^2 \right) + \frac{1}{h} |u(\cdot, t_n; q^\lambda) - g(\cdot, t_n)|^2 \right] \mathrm{d}x \right|_{\lambda=0} \geqslant 0 \tag{2.10}$$

where $u(x, t; q^\lambda)$ is the solution of (1.1)–(1.3) corresponding to

$$\tilde{q} = \begin{cases} \dfrac{t - t_{n-1}}{h} q^\lambda(x) + \dfrac{t_n - t}{h} q_{n-1}(x), & t_{n-1} \leqslant t \leqslant t_n, \\ \dfrac{t - t_{k-1}}{h} q_k(x) + \dfrac{t_k - t}{h} q_{k-1}(x), & t_{k-1} \leqslant t \leqslant t_k, 1 \leqslant k \leqslant n - 1, \end{cases}$$

setting

$$\xi(x, t) = \left. \frac{\mathrm{d}u(x, t; q^\lambda)}{\mathrm{d}\lambda} \right|_{\lambda=0},$$

inequality (2.10) is transformed into

$$\sigma \int_0^1 \left[ \frac{q_n - q_{n-1}}{h}(\omega - q_n) + \nabla q_n \cdot \nabla(\omega - q_n) \right] \mathrm{d}x + \frac{1}{h} \int_0^1 (u^h(x, t_n) - g(x, t_n)) \xi(x, t_n) \, \mathrm{d}x \geqslant 0. \tag{2.11}$$

By direct differentiation with respect to $\lambda$ on both sides of (1.1)–(1.3), it can be verified that $\xi(x, t)$ is the solution of the following parabolic equation:

$$\begin{aligned} & \mathcal{L}\xi \equiv \xi_t - \xi_{xx} + q^h(x, t)\xi = \frac{t - t_{n-1}}{h}(q_n - \omega) u^h, \qquad (x, t) \in (0, 1) \times [t_{n-1}, t_n] \\ & \xi_x(0, t) = \xi_x(1, t) = 0, \\ & \xi(x, t_{n-1}) = 0. \end{aligned} \tag{2.12}$$

Suppose $v^h(x,t)$ is the solution of the following problem:

$$\begin{aligned}
&\mathcal{L}^* v^h \equiv -v_t^h - v_{xx}^h + q^h(x,t)v^h = 0, \qquad (x,t) \in (0,1)\times[t_{n-1},t_n]\\
&v_x^h(0,t) = v_x^h(1,t) = 0,\\
&v^h(x,t_n) = u^h(x,t_n) - g(x,t_n)
\end{aligned} \tag{2.13}$$

where $\mathcal{L}^*$ is the adjoint operator of the operator $\mathcal{L}$. From (2.12) and (2.13) we have

$$\begin{aligned}
0 &= \int_{t_{n-1}}^{t_n}\int_0^1 \xi\mathcal{L}^* v^h \,\mathrm{d}x\,\mathrm{d}t\\
&= -\int_0^1 \xi(x,t_n)[u^h(x,t_n) - g(x,t_n)]\,\mathrm{d}x + \int_{t_{n-1}}^{t_n}\int_0^1 v^h\mathcal{L}\xi\,\mathrm{d}x\,\mathrm{d}t\\
&= -\int_0^1 \xi(x,t_n)[u^h(x,t_n) - g(x,t_n)]\,\mathrm{d}x + \int_{t_{n-1}}^{t_n}\int_0^1 \frac{t-t_{n-1}}{h}(q_n-\omega)u^h v^h\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned} \tag{2.14}$$

Combining (2.11) and (2.14), one can easily obtain that

$$\sigma\int_0^1\left[\frac{q_n-q_{n-1}}{h}(\omega-q_n) + \nabla q_n\cdot\nabla(\omega-q_n)\right]\mathrm{d}x + \frac{1}{h}\int_{t_{n-1}}^{t_n}\int_0^1 \frac{t-t_{n-1}}{h}(q_n-\omega)u^h v^h\,\mathrm{d}x\,\mathrm{d}t \geqslant 0$$

for any $\omega \in \mathcal{A}$.

This completes the proof of theorem 2.2. □

## 3. Some uniform estimates

We will derive some uniform estimates for the sequence of discrete optimal controls $q_0, q_1, \ldots, q_N$ and the discrete reconstruction of unknown coefficient $q^h(x,t)$ as $h \to 0$.

Throughout this paper, $C$ will be denoted the different constants which are independent of parameters $h$ and $\sigma$.

**Lemma 3.1.** *Let $u^h(x,t)$ be the solution of the following problem:*

$$u_t - u_{xx} + \tilde{q}(x,t)u = 0, \qquad (x,t)\in Q \tag{3.1}$$

$$u_x(0,t) = u_x(1,t) = 0 \tag{3.2}$$

$$u(x,0) = u_0(x) \tag{3.3}$$

*with $\tilde{q}(x,t) = q^h(x,t)$. Then there exists a constant C, such that*

$$\|u^h\|_{L^\infty(Q)} + \int_0^T\int_0^1 \left(|u_t^h|^2 + |u_{xx}^h|^2\right)\mathrm{d}x\,\mathrm{d}t + \max_{0\leqslant t\leqslant T}\int_0^1 |u_x^h|^2\,\mathrm{d}x \leqslant C. \tag{3.4}$$

The proof of this lemma is standard.

**Lemma 3.2.** *Let $q^\lambda = (1-\lambda)q_n + \lambda q_{n-1}, 0 \leqslant \lambda \leqslant 1$ and $u^\lambda(x,t) = u(x,t;q^\lambda)$ be the solution of (3.1)–(3.3) in $[0,t_n]$ with*

$$\tilde{q} = \begin{cases} \dfrac{t-t_{n-1}}{h}q^\lambda(x) + \dfrac{t_n-t}{h}q_{n-1}(x), & t_{n-1}\leqslant t\leqslant t_n,\\ \dfrac{t-t_{k-1}}{h}q_k(x) + \dfrac{t_k-t}{h}q_{k-1}(x), & t_{k-1}\leqslant t\leqslant t_k, 1\leqslant k\leqslant n-1. \end{cases} \tag{3.5}$$

*Then there exists a constant C, such that*

$$\|u^\lambda\|_{L^\infty((0,1)\times[t_{n-1}-t_n])} \leqslant C, \tag{3.6}$$

*and*

$$\int_{t_{n-1}}^{t_n}\int_0^1 \left(\left|u_t^\lambda\right|^2 + \left|u_{xx}^\lambda\right|^2\right) \mathrm{d}x\,\mathrm{d}t + \max_{0\leqslant t\leqslant T}\int_0^1 \left|u_x^\lambda\right|^2 \mathrm{d}x \leqslant C\int_{t_{n-1}}^{t_n}\int_0^1 |u^h|^2\,\mathrm{d}x\,\mathrm{d}t + Ch. \tag{3.7}$$

**Proof.** Since the proof of (3.6) is similar to that of (3.4), we only prove (3.7).

Let $\omega = u^\lambda - u^h$. Then from (3.1)–(3.3), it can be verified that $\omega$ satisfies

$$\begin{aligned} &\omega_t - \omega_{xx} + \tilde{q}\omega = (q^h - \tilde{q})u^h, \qquad (x,t) \in (0,1)\times[t_{n-1}, t_n] \\ &\omega_x(0,t) = \omega_x(1,t) = 0 \\ &\omega(x, t_{n-1}) = 0. \end{aligned} \tag{3.8}$$

Multiplying equation (3.8) with $\omega_{xx}$ and integrating over $(0,1)\times[t_{n-1}, t]$, $t \in (t_{n-1}, t_n]$, we obtain that

$$\begin{aligned} &\int_{t_{n-1}}^{t}\int_0^1 \omega_{xx}^2\,\mathrm{d}x\,\mathrm{d}t + \frac{1}{2}\int_{t_{n-1}}^{t}\int_0^1 \left(\omega_x^2\right)_t \mathrm{d}x\,\mathrm{d}t \\ &\quad = \int_{t_{n-1}}^{t}\int_0^1 \tilde{q}\omega\omega_{xx}\,\mathrm{d}x\,\mathrm{d}t + \int_{t_{n-1}}^{t}\int_0^1 (\tilde{q} - q^h)u^h\omega_{xx}\,\mathrm{d}x\,\mathrm{d}t. \end{aligned} \tag{3.9}$$

Noting the boundedness of $\tilde{q}$, one can easily obtain that

$$\begin{aligned} &\int_{t_{n-1}}^{t}\int_0^1 \omega_{xx}^2\,\mathrm{d}x\,\mathrm{d}t + \frac{1}{2}\int_0^1 \omega_x^2(x,t)\,\mathrm{d}x \\ &\quad \leqslant \frac{1}{2}\int_{t_{n-1}}^{t}\int_0^1 \omega_{xx}^2\,\mathrm{d}x\,\mathrm{d}t + C\int_{t_{n-1}}^{t}\int_0^1 \omega^2\,\mathrm{d}x\,\mathrm{d}t + C\int_{t_{n-1}}^{t}\int_0^1 |u^h|^2\,\mathrm{d}x\,\mathrm{d}t \end{aligned} \tag{3.10}$$

By noting (3.4) and (3.6), we have

$$\|\omega\|_{L^\infty((0,1)\times[t_{n-1}-t_n])} \leqslant C.$$

Then from (3.10), we get

$$\int_{t_{n-1}}^{t}\int_0^1 \omega_{xx}^2\,\mathrm{d}x\,\mathrm{d}t + \int_0^1 \omega_x^2(x,t)\,\mathrm{d}x \leqslant C\int_{t_{n-1}}^{t}\int_0^1 |u^h|^2\,\mathrm{d}x\,\mathrm{d}t + Ch. \tag{3.11}$$

From (3.11) and (3.4), one can easily obtain the estimate (3.7).

This completes the proof of lemma 3.2. □

**Lemma 3.3.** *Let $v^\lambda(x,t)$ be the solution of the following problem*

$$\begin{aligned} &-v_t^\lambda - v_{xx}^\lambda + \tilde{q}(x,t)v^\lambda = 0, \qquad (x,t) \in (0,1)\times[t_{n-1}, t_n] \\ &v_x^\lambda(0,t) = v_x^\lambda(1,t) = 0 \\ &v^\lambda(x, t_n) = u^\lambda(x, t_n) - g(x, t_n) \end{aligned} \tag{3.12}$$

*where $\tilde{q}(x,t)$ is defined by (3.5). Then there exists a constant C, such that*

$$\|v^\lambda\|_{L^\infty((0,1)\times[t_{n-1}-t_n])} \leqslant C. \tag{3.13}$$

**Proof.** Noting that

$$\|u^\lambda(x,t_n) - g(x,t_n)\|_{L^\infty((0,1)\times[t_{n-1}-t_n])} \leqslant C,$$

one can easily obtain the result by using the extremum principle. □

**Theorem 3.4.** *Let $q_n \in \mathcal{A}$ be an optimal control of problem $Q_n$. Then there exists a constant $C$, such that*

$$\sum_{n=1}^{N}\int_0^1 \frac{|q_n - q_{n-1}|^2}{h}\,\mathrm{d}x + \max_{1\leqslant n\leqslant N}\int_0^1 |\nabla q_n|^2\,\mathrm{d}x \leqslant C\left(1+\frac{h}{\sigma^2}\right). \tag{3.14}$$

**Proof.** By the definition of $q_n$, i.e., $q_n$ is a minimum of $J_n(q)$, we have

$$J_n(q_n) \leqslant J_n(q_{n-1}). \tag{3.15}$$

From (3.15) we have

$$\sigma\int_0^1\left(\frac{|q_n-q_{n-1}|^2}{h} + |\nabla q_n|^2 - |\nabla q_{n-1}|^2\right)\mathrm{d}x$$
$$\leqslant \frac{1}{h}\int_0^1 [|u(x,t_n;q_{n-1}) - g(x,t_n)|^2 - |u(x,t_n;q_n) - g(x,t_n)|^2]\,\mathrm{d}x \tag{3.16}$$

where $u(x,t_n;q_{n-1})$ is the solution of (3.1)–(3.3) corresponding to coefficient

$$\tilde{q} = \begin{cases} q_{n-1}(x), & t_{n-1}\leqslant t\leqslant t_n,\\ \dfrac{t-t_{k-1}}{h}q_k(x) + \dfrac{t_k - t}{h}q_{k-1}(x), & t_{k-1}\leqslant t\leqslant t_k,\ 1\leqslant k\leqslant n-1. \end{cases}$$

Summing up (3.16) from $n=1$ to $k$, we have

$$\sigma\sum_{n=1}^{k}\int_0^1\frac{|q_n-q_{n-1}|^2}{h}\,\mathrm{d}x + \sigma\int_0^1|\nabla q_k|^2\,\mathrm{d}x$$
$$\leqslant \sigma\int_0^1|\nabla q_0|^2\,\mathrm{d}x + \sum_{n=1}^{k}\frac{1}{h}\int_0^1[|u(x,t_n;q_{n-1}) - g(x,t_n)|^2 - |u(x,t_n;q_n)-g(x,t_n)|^2]\,\mathrm{d}x. \tag{3.17}$$

From the definition of $q^\lambda$ and $u^\lambda$, we have

$$|u(x,t_n;q_{n-1}) - g(x,t_n)|^2 - |u(x,t_n;q_n) - g(x,t_n)|^2$$
$$= \int_0^1 \frac{d|u^\lambda(x,t_n) - g(x,t_n)|^2}{\mathrm{d}\lambda}\,\mathrm{d}\lambda$$
$$= 2\int_0^1 (u^\lambda(x,t_n) - g(x,t_n))\frac{\mathrm{d}u^\lambda(x,t_n)}{\mathrm{d}\lambda}\,\mathrm{d}\lambda.$$

By same argument used in theorem 2.2, we obtain that

$$\int_0^1 (u^\lambda(x,t_n) - g(x,t_n))\frac{\mathrm{d}u^\lambda(x,t_n)}{\mathrm{d}\lambda}\,\mathrm{d}x = \int_{t_{n-1}}^{t_n}\int_0^1 \frac{t-t_{n-1}}{h}(q_n - q_{n-1})u^\lambda v^\lambda\,\mathrm{d}x\,\mathrm{d}t. \tag{3.18}$$

Therefore, from lemma 3.1, 3.2 and 3.3 the right side of equality (3.17) can be estimated by

$$\frac{1}{h}\int_0^1[|u(x,t_n;q_{n-1}) - g(x,t_n)|^2 - |u(x,t_n;q_n) - g(x,t_n)|^2]\,\mathrm{d}x$$
$$= 2\int_0^1\mathrm{d}\lambda\int_{t_{n-1}}^{t_n}\int_0^1\frac{t-t_{n-1}}{h}\cdot\frac{q_n-q_{n-1}}{h}u^\lambda v^\lambda\,\mathrm{d}x\,\mathrm{d}t$$
$$\leqslant \frac{\sigma}{2}\int_{t_{n-1}}^{t_n}\int_0^1\frac{|q_n-q_{n-1}|^2}{h^2}\,\mathrm{d}x\,\mathrm{d}t + \frac{C}{\sigma}\int_0^1\mathrm{d}\lambda\int_{t_{n-1}}^{t_n}\int_0^1\frac{|t-t_{n-1}|^2}{h^2}|u^\lambda|^2|v^\lambda|^2\,\mathrm{d}x\,\mathrm{d}t$$
$$\leqslant \frac{\sigma}{2}\int_0^1\frac{|q_n-q_{n-1}|^2}{h}\,\mathrm{d}x + \frac{C}{\sigma}\|u^\lambda v^\lambda\|^2_{L^\infty((0,1)\times[t_{n-1},t_n])}\int_{t_{n-1}}^{t_n}\frac{|t-t_{n-1}|^2}{h^2}\,\mathrm{d}t$$
$$\leqslant \frac{\sigma}{2}\int_0^1\frac{|q_n-q_{n-1}|^2}{h}\,\mathrm{d}x + \frac{Ch}{\sigma}. \tag{3.19}$$

Combining (3.17) and (3.19) one can easily get

$$\sum_{n=1}^{k}\int_0^1 \frac{|q_n - q_{n-1}|^2}{h}\,\mathrm{d}x + \int_0^1 |\nabla q_k|^2\,\mathrm{d}x$$

$$\leqslant 2\int_0^1 |\nabla q_0|^2\,\mathrm{d}x + \frac{Ch}{\sigma^2} \leqslant C\left(1 + \frac{h}{\sigma^2}\right)$$

for $1 \leqslant k \leqslant N$.

This completes the proof of theorem 3.4. □

From theorem 3.4 one can easily obtain the following theorem.

**Theorem 3.5.** *Assume that there exists a constant C such that* $\frac{h}{\sigma^2} \leqslant C$. *Then for* $q^h$ *we have the following estimate*

$$\int_0^T\int_0^1 |q_t^h|^2\,\mathrm{d}x\,\mathrm{d}t + \max_{0\leqslant t\leqslant T}\int_0^1 |\nabla q^h|^2\,\mathrm{d}x \leqslant C. \tag{3.20}$$

**Theorem 3.6.** *There exists a constant C, such that*

$$\|q^h\|_{C^{\frac{1}{2},\frac{1}{4}}(\bar{Q})} \leqslant C. \tag{3.21}$$

**Proof.** From the definition of the admissible set and the estimates in theorem 3.4, we have

$$\max_{0\leqslant t\leqslant T}\|q^h(\cdot,t)\|_{H^1(0,1)} \leqslant C.$$

Applying Soblev's embedding theorem, there exists a constant $C$ such that

$$|q^h(x,t) - q^h(y,t)| \leqslant C|x-y|^{\frac{1}{2}}$$

for any $t \in [0, T]$.

To obtain the $t$-Hö lder estimate for function $q^h(x,t)$, we assume that for any given points $(x,t), (x,s) \in Q$, without loss of generality, the rectangle

$$D = \{(\xi,\tau)|x \leqslant \xi \leqslant x + \sqrt{t-s}, s \leqslant \tau \leqslant t\} \subset Q.$$

Then we have

$$\int_s^t\int_x^{x+\sqrt{t-s}} q_\tau^h(\xi,\tau)\,\mathrm{d}\xi\,\mathrm{d}\tau = \int_x^{x+\sqrt{t-s}} (q^h(\xi,t) - q^h(\xi,s))\,\mathrm{d}\xi$$

$$= (q^h(\hat{x},t) - q^h(\hat{x},s))\sqrt{t-s},$$

where $\hat{x} = x + \theta\sqrt{t-s}, 0 \leqslant \theta \leqslant 1$.

By theorem 3.5 we obtain that

$$|q^h(\hat{x},t) - q^h(\hat{x},s)| = (t-s)^{-\frac{1}{2}}\int_s^t\int_x^{x+\sqrt{t-s}} q_\tau^h(\xi,\tau)\,\mathrm{d}\xi\,\mathrm{d}\tau$$

$$\leqslant (t-s)^{-\frac{1}{2}}\left(\int_s^t\int_x^{x+\sqrt{t-s}} \mathrm{d}\xi\,\mathrm{d}\tau\right)^{\frac{1}{2}} \cdot \left(\int_0^T\int_0^1 \left|q_t^h\right|^2\mathrm{d}x\,\mathrm{d}t\right)^{\frac{1}{2}}$$

$$\leqslant (t-s)^{\frac{3}{4}-\frac{1}{2}}\left(\int_0^T\int_0^1 \left|q_t^h\right|^2\mathrm{d}x\,\mathrm{d}t\right)^{\frac{1}{2}}$$

$$\leqslant C(t-s)^{\frac{1}{4}}.$$

Then we have

$$|q^h(x,t) - q^h(x,s)| \leqslant |q^h(x,t) - q^h(\hat{x},t)| + |q^h(\hat{x},t) - q^h(\hat{x},s)| + |q^h(\hat{x},s) - q^h(x,s)|$$

$$\leqslant C(t-s)^{\frac{1}{4}}.$$

This completes the proof of theorem 3.6. □

## 4. Asymptotic behavior of $q^h(x,t)$ as $h \to 0$

In this section, we will discuss the asymptotic behavior of the discrete reconstruction, $q^h(x,t)$, of unknown coefficient as $h \to 0$.

Let

$$\tilde{\mathcal{A}} = \{q(x,t) | 0 < \alpha_0 \leqslant q \leqslant \alpha_1, q \in H^1(Q) \cap L^\infty([0,T], H^1(0,1))\},$$

and

$$\bar{q}^h = \begin{cases} q_0(x), & t = t_0, \\ q_k(x), & t_{k-1} < t \leqslant t_k, 1 \leqslant k \leqslant n. \end{cases}$$

From the definition of $\bar{q}^h$, one can easily get

$$\max_{0 \leqslant t \leqslant T} \int_0^1 |\nabla \bar{q}^h|^2 \, \mathrm{d}x \leqslant C.$$

From the estimates in theorem 3.4, 3.5 and 3.6, we have the following convergence results.

**Theorem 4.1.** *Assume that there exists a constant $C$ such that $\frac{h}{\sigma^2} \leqslant C$. Then as $h \to 0$ there exist a subsequence of $q^h(x,t)$ and a function $q(x,t) \in \tilde{\mathcal{A}}$, such that*

$$\begin{aligned} &q^h \to q, && \text{weakly in } H^1(Q), \\ &q^h \to q, && \text{in } C(\bar{Q}), \\ &\bar{q}^h \to q, && \text{in } L^2(Q), \\ &\nabla \bar{q}^h \to \nabla q, && \text{weakly in } L^2(Q). \end{aligned} \tag{4.1}$$

**Proof.** We need only to prove that $q^h$ and $\bar{q}^h$ converge to the same function.

From theorem 3.4, we obtain

$$\begin{aligned} \int_0^T \int_0^1 |q^h - \bar{q}^h|^2 \, \mathrm{d}x \, \mathrm{d}t &= \sum_{n=1}^N \int_{t_{n-1}}^{t_n} \int_0^1 \frac{(t_n - t)^2}{h^2} (q_n - q_{n-1})^2 \, \mathrm{d}x \, \mathrm{d}t \\ &= \frac{h^2}{3} \sum_{n=1}^N \int_0^1 \frac{|q_n - q_{n-1}|^2}{h} \, \mathrm{d}x \leqslant Ch^2 \end{aligned}$$

This implies the result. □

**Lemma 4.2.** *Let $v^h(x,t)$ be the solution of problem (2.9). Then there exists a constant $C$ which is independent of $h$, such that*

$$\int_{t_{n-1}}^{t_n} \int_0^1 \left| v_t^h \right|^2 \mathrm{d}x \, \mathrm{d}t + \max_{t_{n-1} \leqslant t \leqslant t_n} \int_0^1 \left| v_x^h \right|^2 \mathrm{d}x \leqslant C \tag{4.2}$$

*and*

$$\|v^h\|_{C^{\frac{1}{2},\frac{1}{4}}([0,1]\times[t_{n-1},t_n])} \leqslant C. \tag{4.3}$$

**Proof.** The proof of (4.2) is standard. The estimate (4.3) can be derived by the same argument used in theorem 3.6. □

Function $q(x,t) = \lim_{h\to 0} q^h(x,t)$ is the reconstruction of the unknown coefficient. We call it the limiting optimal control of problem **P**. Now we derive the necessary condition of $q(x,t)$.

**Theorem 4.3.** *Assume that there exists a constant C such that* $\frac{h}{\sigma^2} \leqslant C$*. Let* $q(x,t)$ *be the limiting optimal control and* $u(x,t)$ *be the solution of the following problem:*

$$u_t - u_{xx} + q(x,t)u = 0, \qquad (x,t) \in Q \tag{4.4}$$

$$u_x(0,t) = u_x(1,t) = 0 \tag{4.5}$$

$$u(x,0) = u_0(x). \tag{4.6}$$

*Then, for any* $\omega \in \tilde{\mathcal{A}}$*, we have*

$$\int_Q \left[ q_t(\omega - q) + \nabla q \cdot \nabla(\omega - q) + \frac{1}{2\sigma}(u - g)u(q - \omega) \right] \mathrm{d}x\,\mathrm{d}t \geqslant 0. \tag{4.7}$$

**Proof.** Note that $\|q^h\|_{C^{\frac{1}{2},\frac{1}{4}}(\bar{Q})} \leqslant C$. Without loss of generality, we can assume $\alpha \leqslant \frac{1}{2}$, then Schauder's theory guarantees that

$$\|u^h\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(\bar{Q})} \leqslant C, \tag{4.8}$$

and

$$u^h \to u \qquad \text{in} \qquad C^{2,1}(\bar{Q}).$$

We prove (4.7) for $\omega \in \tilde{\mathcal{A}} \bigcap C^\infty(\bar{Q})$ firstly.

Let $\omega$ be a function in $\tilde{\mathcal{A}} \bigcap C^\infty(\bar{Q})$, then $\omega(x, t_n) \in \mathcal{A}$. Thus from the necessary condition (2.8) of sequence of optimal control, we get

$$\int_0^1 \left[ \frac{q_n(x) - q_{n-1}(x)}{h}(\omega(x,t_n) - q_n(x)) + \nabla q_n(x) \cdot (\nabla\omega(x,t_n) - \nabla q_n(x)) \right.$$
$$\left. + f_n(x)(q_n(x) - \omega(x,t_n)) \right] \mathrm{d}x \geqslant 0, \tag{4.9}$$

where

$$f_n(x) = \frac{1}{\sigma h}\int_{t_{n-1}}^{t_n} \frac{t - t_{n-1}}{h} u^h(x,t)v^h(x,t)\,\mathrm{d}t, \qquad 1 \leqslant n \leqslant N. \tag{4.10}$$

From the definition of $q^h$ and $\bar{q}^h$, it follows that

$$\int_0^1 \int_{t_{n-1}}^{t_n} \left[ q_t^h(x,t)(\omega(x,t_n) - \bar{q}^h(x,t)) + \nabla\bar{q}^h(x,t) \cdot (\nabla\omega(x,t_n) - \nabla\bar{q}^h(x,t)) \right.$$
$$\left. + \frac{t - t_{n-1}}{\sigma h} u^h(x,t)v^h(x,t)(\bar{q}^h(x,t) - \omega(x,t_n)) \right] \mathrm{d}t\,\mathrm{d}x \geqslant 0.$$

Therefore, we obtain

$$\int_0^1 \int_{t_{n-1}}^{t_n} \left[ q_t^h(x,t)(\omega(x,t) - \bar{q}^h(x,t)) + \nabla\bar{q}^h(x,t) \cdot (\nabla\omega(x,t) - \nabla\bar{q}^h(x,t)) \right.$$
$$\left. + \frac{t - t_{n-1}}{\sigma h} u^h(x,t)v^h(x,t)(\bar{q}^h(x,t) - \omega(x,t)) \right] \mathrm{d}t\,\mathrm{d}x \geqslant E_n \tag{4.11}$$

where

$$E_n = \int_0^1 \int_{t_{n-1}}^{t_n} \left[ q_t^h(x,t)(\omega(x,t) - \omega(x,t_n)) + \nabla\bar{q}^h(x,t) \cdot (\nabla\omega(x,t) - \nabla\omega(x,t_n)) \right.$$
$$\left. + \frac{t - t_{n-1}}{\sigma h} u^h(x,t)v^h(x,t)(\omega(x,t_n) - \omega(x,t)) \right] \mathrm{d}t\,\mathrm{d}x$$
$$= I_1 + I_2 + I_3.$$

By the smoothnes‘s of $\omega$, there exists a constant $C$ which is independent of $h$, such that

$$|\omega(x,t)-\omega(x,t_n)| = |\omega_t(x,t_n-\theta(t_n-t))(t-t_n)| \leqslant Ch,$$

where $0\leqslant\theta\leqslant 1$. Then, for item $I_1$, we have the following estimate

$$\begin{aligned}|I_1| &\leqslant Ch\int_0^1\int_{t_{n-1}}^{t_n}\left|q_t^h(x,t)\right|\mathrm{d}x\,\mathrm{d}t\\ &\leqslant Ch\left(\int_0^1\int_{t_{n-1}}^{t_n}\left|q_t^h(x,t)\right|^2\mathrm{d}x\,\mathrm{d}t+\int_0^1\int_{t_{n-1}}^{t_n}\mathrm{d}x\,\mathrm{d}t\right)\\ &\leqslant Ch\int_0^1\int_{t_{n-1}}^{t_n}\left|q_t^h(x,t)\right|^2\mathrm{d}x\,\mathrm{d}t+Ch^2.\end{aligned}$$

So, for $I_2$ we have similar estimate by the same argument of $I_1$. For item $I_3$ we have the following estimate

$$\begin{aligned}|I_3| &\leqslant \frac{Ch}{\sigma}\int_{t_{n-1}}^{t_n}\frac{t-t_{n-1}}{h}\,\mathrm{d}t\\ &\leqslant \frac{Ch^2}{\sigma}\leqslant Ch^{\frac{3}{2}}\frac{\sqrt{h}}{\sigma}\leqslant Ch^{\frac{3}{2}}.\end{aligned}$$

Therefore, for $E_n$, we obtain that

$$|E_n|\leqslant C\big(h^2+h^{\frac{3}{2}}\big)+Ch\int_0^1\int_{t_{n-1}}^{t_n}\left|q_t^h\right|^2\mathrm{d}x\,\mathrm{d}t+Ch^2\int_0^1|\nabla q_n|^2\,\mathrm{d}x.$$

It is easily seen that

$$\sum_{n=1}^{N}|E_n|\leqslant Ch^{\frac{1}{2}}. \tag{4.12}$$

From (4.11) and (4.12), we have

$$\begin{aligned}\sigma\int_0^1\int_0^T &\left[q_t^h(x,t)(\omega(x,t)-\bar q^h(x,t))+\nabla\bar q^h(x,t)\cdot(\nabla\omega(x,t)-\nabla\bar q^h(x,t))\right]\mathrm{d}t\,\mathrm{d}x\\ &+\sum_{n=1}^{N}\int_0^1\int_{t_{n-1}}^{t_n}\frac{t-t_{n-1}}{h}u^h(x,t)v^h(x,t)(\bar q^h(x,t)-\omega(x,t))\,\mathrm{d}t\,\mathrm{d}x\geqslant -Ch^{\frac{1}{2}}.\end{aligned} \tag{4.13}$$

By noting that

$$v^h(x,t_n)=u^h(x,t_n)-g(x,t_n),$$

we obtain

$$\begin{aligned}&|v^h(x,t)-(u^h(x,t)-g(x,t))|\\ &\quad\leqslant|v^h(x,t)-v^h(x,t_n)|+|(u^h(x,t_n)-g(x,t_n))-(u^h(x,t)-g(x,t))|\\ &\quad\leqslant Ch^{\frac{\alpha}{2}},\end{aligned}$$

where we have used (2.1, (4.8) and lemma 4.2.

Hence

$$\begin{aligned}&\left|\sum_{n=1}^{N}\int_0^1\int_{t_{n-1}}^{t_n}\frac{t-t_{n-1}}{h}u^h(x,t)[v^h(x,t)-(u^h(x,t)-g(x,t))](\bar q^h(x,t)-\omega(x,t))\,\mathrm{d}t\,\mathrm{d}x\right|\\ &\quad\leqslant Ch^{\frac{\alpha}{2}}.\end{aligned} \tag{4.14}$$

From (4.13) and (4.14), we obtain that

$$\sigma \int_0^1 \int_0^T \left[q_t^h(x,t)(\omega(x,t) - \bar{q}^h(x,t)) + \nabla \bar{q}^h(x,t)\cdot(\nabla\omega(x,t) - \nabla\bar{q}^h(x,t))\right] \mathrm{d}t\,\mathrm{d}x$$
$$+ \sum_{n=1}^{N} \int_0^1 \int_{t_{n-1}}^{t_n} \frac{t - t_{n-1}}{h}(u^h(x,t) - g(x,t))u^h(x,t)(\bar{q}^h(x,t) - \omega(x,t))\,\mathrm{d}t\,\mathrm{d}x$$
$$\geqslant -Ch^{\frac{\alpha}{2}}. \tag{4.15}$$

Note that

$$\int_{t_{n-1}}^{t_n} \left(\frac{t - t_{n-1}}{h} - \frac{1}{2}\right) \mathrm{d}t = 0.$$

By the direct calculation, we have

$$\int_{t_{n-1}}^{t_n} \frac{t - t_{n-1}}{h} f^h(t)\,\mathrm{d}t - \frac{1}{2}\int_{t_{n-1}}^{t_n} f^h(t)\,\mathrm{d}t$$
$$= \int_{t_{n-1}}^{t_n} \left(\frac{t - t_{n-1}}{h} - \frac{1}{2}\right)(f^h(t) - f^h(t_n))\,\mathrm{d}t, \tag{4.16}$$

where

$$f^h(t) = \int_0^1 (u^h(x,t) - g(x,t))u^h(x,t)(\bar{q}^h(x,t) - \omega(x,t))\,\mathrm{d}x.$$

It is easily seen that

$$f^h(t) \in C^{\frac{\alpha}{2}}[t_{n-1}, t_n].$$

Hence we have from (4.16)

$$\left| \int_{t_{n-1}}^{t_n} \frac{t - t_{n-1}}{h} f^h(t)\,\mathrm{d}t - \frac{1}{2}\int_{t_{n-1}}^{t_n} f^h(t)\,\mathrm{d}t \right| \leqslant \|f^h\|_{C^{\frac{\alpha}{2}}[t_{n-1},t_n]} h^{1+\frac{\alpha}{2}}. \tag{4.17}$$

From (4.17) we obtain that

$$\left| \int_0^1 \int_{t_{n-1}}^{t_n} \frac{t - t_{n-1}}{h}(u^h(x,t) - g(x,t))u^h(x,t)(\bar{q}^h(x,t) - \omega(x,t))\,\mathrm{d}t\,\mathrm{d}x \right.$$
$$\left. - \frac{1}{2}\int_0^1 \int_{t_{n-1}}^{t_n} (u^h(x,t) - g(x,t))u^h(x,t)(\bar{q}^h(x,t) - \omega(x,t))\,\mathrm{d}t\,\mathrm{d}x \right| \leqslant Ch^{1+\frac{\alpha}{2}}. \tag{4.18}$$

Summing up (4.18) from $n = 1$ to $N$, we have

$$\left| \sum_{n=1}^{N} \int_0^1 \int_{t_{n-1}}^{t_n} \frac{t - t_{n-1}}{h}(u^h(x,t) - g(x,t))u^h(x,t)(\bar{q}^h(x,t) - \omega(x,t))\,\mathrm{d}t\,\mathrm{d}x \right.$$
$$\left. - \frac{1}{2}\int_Q (u^h(x,t) - g(x,t))u^h(x,t)(\bar{q}^h(x,t) - \omega(x,t))\,\mathrm{d}t\,\mathrm{d}x \right| \leqslant Ch^{\frac{\alpha}{2}}. \tag{4.19}$$

Combining (4.15) and (4.19), one can easily obtain

$$\int_Q \left[q_t^h(\omega - \bar{q}^h) + \nabla\bar{q}^h\cdot\nabla(\omega - \bar{q}^h) + \frac{1}{2\sigma}(u^h - g)u^h(\bar{q}^h - \omega)\right] \mathrm{d}x\,\mathrm{d}t \geqslant -Ch^{\frac{\alpha}{2}}. \tag{4.20}$$

Letting $h \to 0$, we deduce that, from theorem 4.1 and (4.20)

$$\int_Q \left[ q_t(\omega - q) + \nabla q \cdot \nabla \omega + \frac{1}{2\sigma}(u-g)u(q-\omega) \right] \mathrm{d}x\,\mathrm{d}t - \limsup_{h\to 0} \int_Q |\nabla \bar{q}^h|^2 \,\mathrm{d}x\,\mathrm{d}t \geqslant 0. \tag{4.21}$$

By the property of weak convergence, we have

$$\liminf_{h\to 0} \int_Q |\nabla \bar{q}^h|^2 \,\mathrm{d}x\,\mathrm{d}t \geqslant \int_Q |\nabla q|^2 \,\mathrm{d}x\,\mathrm{d}t.$$

Then from (4.21), we deduce that

$$\int_Q \left[ q_t(\omega - q) + \nabla q \cdot \nabla(\omega - q) + \frac{1}{2\sigma}(u-g)u(q-\omega) \right] \mathrm{d}x\,\mathrm{d}t \geqslant 0 \tag{4.22}$$

for any $\omega \in \tilde{\mathcal{A}} \bigcap C^\infty(\bar{Q})$.

The necessary condition (4.22) remains true for any $\omega \in \tilde{\mathcal{A}}$ by the approximation argument.

This completes the proof of theorem 4.3. □

**Corollary 4.4.** *Let $q(x,t)$ be the limiting optimal control and $u(x,t)$ be the solution of (4.4)–(4.6). Then, for any $\omega \in \tilde{\mathcal{A}}$, we have*

$$\int_0^s \int_0^1 \left[ q_t(\omega - q) + \nabla q \cdot \nabla(\omega - q) + \frac{1}{2\sigma}(u-g)u(q-\omega) \right] \mathrm{d}x\,\mathrm{d}t \geqslant 0, \tag{4.23}$$

*for any $s \in [0, T]$.*

**Proof.** Let $\delta > 0$ and $\eta_\delta \in C^1[0,T]$ be a cut-off function such that

$$\eta_\delta(t) = \begin{cases} 1, & 0 \leqslant t \leqslant s \\ 0, & s+\delta \leqslant t \leqslant T \end{cases}.$$

Note that $\tilde{\mathcal{A}}$ is a convex set, for any $\omega \in \tilde{\mathcal{A}}$,

$$\tilde{\omega} = q + \eta_\delta(\omega - q) \in \tilde{\mathcal{A}}.$$

From theorem 4.3 we have

$$\int_Q \left[ q_t(\tilde{\omega} - q) + \nabla q \cdot \nabla(\tilde{\omega} - q) + \frac{1}{2\sigma}(u-g)u(q-\tilde{\omega}) \right] \mathrm{d}x\,\mathrm{d}t \geqslant 0.$$

Hence

$$\int_Q \left[ q_t(\omega - q) + \nabla q \cdot \nabla(\omega - q) + \frac{1}{2\sigma}(u-g)u(q-\omega) \right] \eta_\delta(t)\,\mathrm{d}x\,\mathrm{d}t \geqslant 0.$$

Letting $\delta \to 0$, we obtain the result. □

## 5. Stability and uniqueness

Now we derive the stability and uniqueness of limiting optimal controls in the sense of $L^2$ norm.

**Theorem 4.1.** *Suppose that $q_0(x), \bar{q}_0(x), g(x,t), \bar{g}(x,t)$ are given functions, $q_0, \bar{q}_0 \in \mathcal{A}$ and $g, \bar{g}$ satisfy conditions (2.1) and (2.2). Let $q(x,t), \bar{q}(x,t)$ be the limiting optimal controls corresponding to $(q_0, g), (\bar{q}_0, \bar{g})$, respectively. Then there exists a constant $C$ such that*

$$\max_{t\in[0,T]} \int_0^1 |q - \bar{q}|^2 \,\mathrm{d}x + \int_Q |\nabla(q-\bar{q})|^2 \,\mathrm{d}x\,\mathrm{d}t \leqslant \frac{C}{\sigma}\left( \int_0^1 |q_0 - \bar{q}_0|^2 \,\mathrm{d}x + \int_Q |g-\bar{g}|^2 \,\mathrm{d}x\,\mathrm{d}t \right). \tag{5.1}$$

**Proof.** Let $u(x,t), \bar{u}(x,t)$ be the solution of (4.4)–(4.6) corresponding to $q(x,t), \bar{q}(x,t)$, respectively. It can be easily verified that $W = u - \bar{u}$ satisfies the following equation

$$W_t - W_{xx} + q(x,t)W = (\bar{q} - q)\bar{u}, \qquad (x,t) \in Q \tag{5.2}$$

$$W_x(0,t) = W_x(1,t) \tag{5.3}$$

$$W(x,0) = 0. \tag{5.4}$$

By $L^2$ theory for parabolic equation (see [13]), we deduce that

$$\|W\|_{L^2(Q_s)} \leqslant C\|\bar{q} - q\|_{L^2(Q_s)}, \tag{5.5}$$

where $s \in [0,T]$, $Q_s = (0,1) \times [0,s]$.

From corollary 4.4, we have, for any $s \in [0,T]$

$$\int_0^s \int_0^1 \left[ q_t(\bar{q} - q) + \nabla q \cdot \nabla(\bar{q} - q) + \frac{1}{2\sigma}(u - g)u(q - \bar{q}) \right] \mathrm{d}x\,\mathrm{d}t \geqslant 0, \tag{5.6}$$

and

$$\int_0^s \int_0^1 \left[ \bar{q}_t(q - \bar{q}) + \nabla \bar{q} \cdot \nabla(q - \bar{q}) + \frac{1}{2\sigma}(\bar{u} - \bar{g})\bar{u}(\bar{q} - q) \right] \mathrm{d}x\,\mathrm{d}t \geqslant 0. \tag{5.7}$$

Hence

$$\int_0^s \int_0^1 [(q - \bar{q})(q - \bar{q})_t + |\nabla(q - \bar{q})|^2]\,\mathrm{d}x\,\mathrm{d}t \leqslant \frac{1}{2\sigma} \int_0^s \int_0^1 (q - \bar{q})[(u - g)u - (\bar{u} - \bar{g})\bar{u}]\,\mathrm{d}x\,\mathrm{d}t. \tag{5.8}$$

Therefore, we obtain

$$\sigma \int_0^1 (q - \bar{q})^2(x,s)\,\mathrm{d}x + 2\sigma \int_0^s \int_0^1 |\nabla(q - \bar{q})|^2\,\mathrm{d}x\,\mathrm{d}t \leqslant \sigma \int_0^1 (q_0 - \bar{q}_0)^2\,\mathrm{d}x + \int_0^s \int_0^1 (q - \bar{q})^2\,\mathrm{d}x\,\mathrm{d}t + \int_0^s \int_0^1 [(u - g)u - (\bar{u} - \bar{g})\bar{u}]^2\,\mathrm{d}x\,\mathrm{d}t. \tag{5.9}$$

Note that

$$\begin{aligned}
[(u - g)u - (\bar{u} - \bar{g})\bar{u}]^2 &= [(u^2 - \bar{u}^2) + (\bar{u}\bar{g} - ug)]^2 \\
&\leqslant C[(u^2 - \bar{u}^2)^2 + (\bar{u}\bar{g} - \bar{u}g + \bar{u}g - ug)^2] \\
&\leqslant C[(u + \bar{u})^2(u - \bar{u})^2 + \bar{u}^2(\bar{g} - g)^2 + g^2(\bar{u} - u)^2] \\
&\leqslant C[(g - \bar{g})^2 + (u - \bar{u})^2].
\end{aligned}$$

Hence

$$\int_0^s \int_0^1 [(u - g)u - (\bar{u} - \bar{g})\bar{u}]^2\,\mathrm{d}x\,\mathrm{d}t \leqslant C\big(\|g - \bar{g}\|^2_{L^2(Q_s)} + \|u - \bar{u}\|^2_{L^2(Q_s)}\big). \tag{5.10}$$

From (5.5), (5.10) and (5.9), we have

$$\sigma \int_0^1 (q - \bar{q})^2(x,s)\,\mathrm{d}x + 2\sigma \int_0^s \int_0^1 |\nabla(q - \bar{q})|^2\,\mathrm{d}x\,\mathrm{d}t \leqslant \sigma \int_0^1 (q_0 - \bar{q}_0)^2\,\mathrm{d}x + C\int_0^s \int_0^1 (q - \bar{q})^2\,\mathrm{d}x\,\mathrm{d}t + C\int_0^s \int_0^1 (g - \bar{g})^2\,\mathrm{d}x\,\mathrm{d}t. \tag{5.11}$$

By Gronwall's inequality we obtain that

$$\int_0^1 (q-\bar{q})^2(x,s)\,\mathrm{d}x + \int_0^s \int_0^1 |\nabla(q-\bar{q})|^2\,\mathrm{d}x\,\mathrm{d}t \leqslant \frac{C}{\sigma}\left(\int_0^1 (q_0-\bar{q}_0)^2\,\mathrm{d}x + \int_0^s \int_0^1 (g-\bar{g})^2\,\mathrm{d}x\,\mathrm{d}t\right)$$

for any $s \in [0, T]$.

This completes the proof. □

**Remark 5.1.** It should be mentioned that the regularization parameter plays a major role in the numerical simulation of ill-posed problems. From theorem 5.1 we can obtain that if there exists a constant $\delta$ such that

$$\|q_0-\bar{q}_0\| \leqslant \delta, \qquad \|g-\bar{g}\| \leqslant \delta, \quad \text{and} \quad \frac{\delta^2}{\sigma} \to 0,$$

then the reconstructed limiting optimal control is unique and stable. In general, for ill-posed problems a convergence result can be obtained when the regularization parameter depends in a proper way on data error which goes to zero. Moreover, a rate of convergence can also be derived under some additional assumptions. For more detailed discussion on the regularization parameter, we refer the readers to the references, e.g., in [5, 11, 12].

## 6. Concluding remarks

The inverse problem of identifying the radiative coefficient in heat conduction problem from some additional conditions is very important in some engineering texts and many industrial applications. The difficulty is due to the lack of conventional stability and to nonlinearity and nonconvexity.

In this paper, we solve the inverse problem **P** of recovering the radiative coefficient $q(x,t)$ in the following heat conduction equation

$$u_t - u_{xx} + q(x,t)u = 0$$

in an optimal control framework. Such problem is a natural extension of that in [31]. The unknown coefficient in [31] is purely space dependent, while in this paper it not only depends on the space variable $x$, but also depends on the time $t$, which may occur in the case that the property of heat conductor varies with space and time. Based on the idea in [31], we transform problem **P** into a sequence of inverse problems $P_n, n = 1, 2, \ldots, N$, which are similar to the problem in [31], i.e., the unknown coefficient is purely space dependent. The existence, as well as the necessary condition of the minimizer $q_n(x)$ for problem $P_n$ is established. By the obtained $q_n$, we define the discrete reconstruction $q^h(x,t)$ and prove that there exists a subsequence of $q^h(x,t)$ converging to a function $q(x,t)$ which is called the limiting optimal control of problem **P**. Finally, we obtain the uniqueness and stability of $q(x,t)$ in the sense of $L^2$ norm.

The paper focuses on the theoretical analysis of the 1D inverse problem. For the multidimensional case, i.e., the determination of $q(x,t)$ in the following equation:

$$u_t - \Delta u + q(x,t)u = 0, \qquad (x,t) \in Q = \Omega \times (0, T],$$

where $\Omega \in \mathscr{R}^m (m \geqslant 1)$ is a given bounded domain, we believe the method used in this paper is also applicable.

The mathematical model discussed in the paper is linear. Next, we will consider the nonlinear case, e.g., the determination of a pair $(q, u)$ in the following nonlinear parabolic equation

$$u_t - \triangle u + q(x, t) f(u) = 0, \qquad (x, t) \in Q = (0, 1) \times (0, T]$$

from the over-specified data $u(x, t) = g(x, t)$ (see [33], where the purely space dependent case, i.e., $q = q(x)$, has been considered carefully).

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